

Parameterized Integer Quadratic Programming: Variables and Coefficients

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Abstract

In the INTEGER QUADRATIC PROGRAMMING problem input is an $n \times n$ integer matrix Q , an $m \times n$ integer matrix A and an m -dimensional integer vector b . The task is to find a vector $x \in \mathbb{Z}^n$ minimizing $x^T Q x$, subject to $Ax \leq b$. We give a fixed parameter tractable algorithm for INTEGER QUADRATIC PROGRAMMING parameterized by $n + \alpha$, assuming that an optimal solution to the input instance exists. Here α is the largest absolute value of an entry of Q and A . As an application of our main result we show that OPTIMAL LINEAR ARRANGEMENT is fixed parameter tractable parameterized by the size of the smallest vertex cover of the input graph. This resolves an open problem from the recent monograph by Downey and Fellows.

1 Introduction

While LINEAR PROGRAMMING is famously polynomial time solvable [15], most generalizations are not. In particular, requiring the variables to take integer values gives us the INTEGER LINEAR PROGRAMMING problem, which is easily seen to be NP-hard. On the other hand, integer linear programs (ILPs) with few variables can be solved efficiently. The celebrated algorithm of Lenstra [12] solves ILPs with n variables in time $f(n)L^{O(1)}$ where f is a (doubly exponential) function depending only on n and L is the total number of bits required to encode the input integer linear program. In terms of parameterized complexity this means that INTEGER LINEAR PROGRAMMING is *fixed parameter tractable* (FPT) when parameterized by the number n of variables to the input ILP. In parameterized complexity input instances come with a *parameter* k , and an algorithm is called fixed parameter tractable if it solves instances of size L with parameter k in time $f(k)L^{O(1)}$ for some function f depending only on k . For an introduction to parameterized complexity we refer to the recent monograph of Downey and Fellows [6], as well as the textbook by Cygan et al. [5].

Following the algorithm of Lenstra [12] there has been a significant amount of research into parameterized algorithms for INTEGER LINEAR PROGRAMMING, as well as generalizations of the problem. Highlights include the algorithms for INTEGER LINEAR PROGRAMMING with improved dependence on n by Kannan [13], Clarkson [4] and Frank and Tardos [10]. Khachiyan and Porkolab [14] generalized the FPT algorithm of Lenstra to find $\min\{y_n : (y_1, \dots, y_n) \in Y \cap \mathbb{Z}^n\}$ where Y is a *convex* set defined by polynomial inequalities of fixed constant degree $d \geq 2$. For every fixed d the algorithm of Khachiyan and Porkolab [14] is FPT when parameterized by the number n of variables.

Surprisingly little is known about the parameterized complexity of generalizations of INTEGER LINEAR PROGRAMMING to optimization of possibly non-convex functions over possibly non-convex domains. Perhaps the simplest such generalization is the INTEGER QUADRATIC PROGRAMMING problem. Here input is a $n \times n$ integer matrix Q , an $m \times n$ integer matrix A

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and an m -dimensional integer vector b . The task is to find a vector $x \in \mathbb{Z}^n$ minimizing $x^T Q x$, subject to $Ax \leq b$. Thus, in this problem, the domain is convex, but the objective function might not be.

To the best of our knowledge it is quite possible that INTEGER QUADRATIC PROGRAMMING is fixed parameter tractable when parameterized by the number n of variables. Such a result would be a major breakthrough, as it would significantly generalize the classic FPT algorithms for INTEGER LINEAR PROGRAMMING. In this paper we take a more modest approach to INTEGER QUADRATIC PROGRAMMING, and consider the problem when parameterized by the number n of variables and the largest absolute value α of the entries in the matrices Q and A . Our main result is an algorithm for INTEGER QUADRATIC PROGRAMMING with running time $f(n, \alpha)L^{O(1)}$, demonstrating that the problem is fixed parameter tractable when parameterized by the number of variables and the largest coefficient appearing in the objective function and in the constraints. Our algorithm has a caveat: it only finds an optimal solution if there is one. For *unbounded* quadratic integer programs (QIPs), that is QIPs where there exist feasible solutions with arbitrarily small value of the objective function, the algorithm outputs a feasible solution, but fails to detect that the QIP is in fact unbounded. It is possible to extend our methods to also detect that the input QIP is unbounded, however we do not do it in this article.

On one hand INTEGER QUADRATIC PROGRAMMING is a more general problem than INTEGER LINEAR PROGRAMMING. On the other hand the parameterization by variables and coefficients is a much stronger parameterization than parameterizing just by the number n of variables. By making the largest entry α of Q and A a parameter we allow the running time of our algorithms to depend in arbitrary ways on essentially all of the input. The only reason that designing an FPT algorithm for this parameterization is non-trivial is that the entries in the vector b may be arbitrarily large compared to the parameters n and α . This makes the number of possible assignments to the variables much too large to enumerate all assignments by brute force. Indeed, despite being quite restricted our algorithm for INTEGER QUADRATIC PROGRAMMING allows us to show fixed parameter tractability of a problem whose parameterized complexity was unknown prior to this work. More concretely we use the new algorithm for INTEGER QUADRATIC PROGRAMMING to prove that OPTIMAL LINEAR ARRANGEMENT parameterized by the size of the smallest vertex cover of the input graph is fixed parameter tractable.

In the OPTIMAL LINEAR ARRANGEMENT problem we are given as input an undirected graph G on n vertices. The task is to find a permutation $\sigma : V(G) \rightarrow \{1, \dots, n\}$ minimizing the *cost* of σ . Here the cost of a permutation σ is $val(\sigma, G) = \sum_{uv \in E(G)} |\sigma(u) - \sigma(v)|$. The problem was shown to be NP-complete already in the 70's [11], admits a factor $O(\sqrt{\log n} \log \log n)$ approximation algorithm [2, 7], but no admits no polynomial time approximation scheme, assuming plausible complexity-theoretic assumptions [1].

We consider OPTIMAL LINEAR ARRANGEMENT parameterized by the size of the smallest *vertex cover* of the input graph G . A *vertex cover* of a graph G is a vertex set C such that every edge in G has at least one endpoint in C . When OPTIMAL LINEAR ARRANGEMENT is parameterized by the vertex cover number of the input graph, an integer parameter k is also given as input together with G and n . An FPT algorithm is allowed to run in time $f(k)n^{O(1)}$ and only has to provide an optimal layout σ of G if there exists a vertex cover in G of size at most k . We remark that one can compute a vertex cover of size k , if it exists, in time $O(1.2748^k + n^{O(1)})$ [3]. Hence, when designing an algorithm for OPTIMAL LINEAR ARRANGEMENT parameterized by vertex cover we may just as well assume that a vertex cover C of G of size at most k is given as input.

The parameterized complexity of OPTIMAL LINEAR ARRANGEMENT parameterized by vertex cover was first posed as an open problem by Fellows et al. [9]. Fellows et al. [9] showed that a number of well-studied graph layout problems, such as BANDWIDTH and CUTWIDTH can be shown to be FPT when parameterized by vertex cover, by reducing the problems to INTEGER

LINEAR PROGRAMMING parameterized by the number of variables. For the most natural formalization of OPTIMAL LINEAR ARRANGEMENT as an integer program the objective function is quadratic (and not necessarily convex), and therefore the above approach fails.

Motivated by the lack of progress on this problem, Fellows et al. [8] recently showed an FPT approximation scheme for OPTIMAL LINEAR ARRANGEMENT parameterized by vertex cover. In particular they gave an algorithm that given as input a graph G with a vertex cover of size at most k and a rational $\epsilon > 0$, produces in time $f(k, \epsilon)n^{O(1)}$ a layout σ with cost at most a factor $(1 + \epsilon)$ larger than the optimum. Fellows et al. [8] re-state the parameterized complexity of OPTIMAL LINEAR ARRANGEMENT parameterized by vertex cover as an open problem. Finally, the problem was re-stated as an open problem in the recent monograph of Downey and Fellows [6]. Interestingly, Downey and Fellows motivate the study of this problem as follows.

“Our enthusiasm for this concrete problem is based on its connection to INTEGER LINEAR PROGRAMMING. The problem above is easily reducible to a restricted form of INTEGER QUADRATIC PROGRAMMING which may well be FPT”.

We give an FPT algorithm for OPTIMAL LINEAR ARRANGEMENT parameterized by vertex cover, resolving the open problem of [6, 8, 9]. Our algorithm for OPTIMAL LINEAR ARRANGEMENT works by directly applying the new algorithm for INTEGER QUADRATIC PROGRAMMING to the most natural formulation of OPTIMAL LINEAR ARRANGEMENT on graphs with a small vertex cover as an integer quadratic program, confirming the intuition of Downey and Fellows [6].

Preliminaries and notation.

In Section 2 lower case letters denote vectors and scalars, while upper case letters denote matrices. All vectors are column vectors. For an integer $p \geq 2$, the ℓ_p norm of an n -dimensional vector $v = [v_1, v_2, \dots, v_n]$ is denoted by $|v|_p$ and is defined to be $|v|_p = (\sqrt[p]{v_1^p + v_2^p \dots v_n^p})^{1/p}$. The ℓ_1 norm of v is $|v|_1 = |v_1| + |v_2| + \dots + |v_n|$, while the ℓ_∞ norm of v is $|v|_\infty = \max(|v_1|, |v_2|, \dots, |v_n|)$.

2 Algorithm for Integer Quadratic Programming

We consider the following problem, called INTEGER QUADRATIC PROGRAMMING. Input consists of an $n \times n$ integer symmetric matrix Q , an $m \times n$ integer matrix A and m -dimensional integer vector b . The task is to find an optimal solution x^* to the following optimization problem.

$$\begin{aligned} & \text{Minimize } x^T Q x \\ & \text{subject to: } Ax \leq b \\ & \quad x \in \mathbb{Z}^n. \end{aligned} \tag{1}$$

A vector $x \in \mathbb{Z}^n$ that satisfies the constraints $Ax \leq b$ is called a *feasible solution* to the IQP (1). Given an input on the form (1) there are three possible scenarios. A possible scenario is that there are no feasible solutions, in which case this is what the an algorithm for INTEGER QUADRATIC PROGRAMMING should report. Another possibility is that for every integer β there exists some feasible solution x such that $x^T Q x \leq \beta$. In that case the algorithm should report that the IQP is *unbounded*. Finally, it could be that there exist feasible solutions, and that the minimum value of $x^T Q x$ over the set of all feasible x is well defined. This is the most interesting case, and in this case the algorithm should output a feasible x such that $x^T Q x$ is minimized.

Note that the requirement that Q is symmetric can easily be avoided by replacing Q by $Q + Q^T$. This operation does not change the set of optimal solutions, since it multiplies the objective function value of every solution by 2. We will denote by a_i^T the i 'th row of the matrix A , and by b_i the i 'th entry of the vector b . Thus, $Ax \leq b$ means that $a_i^T x \leq b_i$ for all i . The maximum absolute value of an entry of A and Q is denoted by α . Using a pair of inequalities

one can encode equality constraints. It is useful to rewrite the IQP (1) to separate out the equality constraints explicitly, obtaining the following equivalent form.

$$\begin{aligned}
& \text{Minimize } x^T Q x \\
& \text{subject to: } Ax \leq b \\
& \quad \quad \quad Cx = d \\
& \quad \quad \quad x \in \mathbb{Z}^n.
\end{aligned} \tag{2}$$

Here C is an integer matrix and d is an integer vector. If input is given on the form (2), then we still use α to denote the maximum value of an entry of A and Q . The IQP (2) could be generalized by changing the objective function from $x^T Q x$ to $x^T Q x + q^T x$ for some n -dimensional vector q also given as input. This generalization can be incorporated in the original formulation (2) at the cost of introducing a new variable \hat{x} , adding the constraint $\hat{x} = 1$ to the system $Cx = d$ and adding $[0, q]$ as the row corresponding to the new variable \hat{x} in Q .

We will denote by Δ the maximum absolute value of the determinant of a square submatrix of C . We may assume without loss of generality that the rows of C are linearly independent; otherwise we may in polynomial time either conclude that the IQP has no feasible solutions, or remove one of the equality constraints in the system $Cx = d$ without changing the set of feasible solutions. Thus C has at most n rows. If the IQP (2) is obtained from (1) by replacing constraints $a_i^T x \leq b_i$, $-a_i^T x \leq -b_i$ with $a_i^T x = b_i$, the maximum entry of C is also upper bounded by α and then we have $\Delta \leq n! \cdot \alpha^n$. The next simple observation shows that we can in polynomial time reduce the number of constraints to a function of n and α .

Lemma 1. *There is a polynomial time algorithm that given as input the matrix A and vector b outputs an $m' \times n$ submatrix A' of A and vector b' such that $m' \leq (2\alpha + 1)^n$ and for every $x \in \mathbb{Z}^n$, $Ax \leq b$ if and only if $A'x \leq b'$.*

Proof. Suppose A has more than $(2\alpha + 1)^n$ rows. Then, by the pigeon hole principle the system $Ax \leq b$ has two rows $a_i^T x \leq b_i$ and $a_j^T x \leq b_j$ where $i \neq j$ but $a_i = a_j$. Without loss of generality $b_i \leq b_j$, and then any $x \in \mathbb{Z}^n$ such that $a_i^T x \leq b_i$ satisfies $a_j^T x \leq b_j$. Thus we can safely remove the inequality $a_j^T x \leq b_j$ from the system, and the lemma follows. \square

In the following we will assume that the input is on the form (2). We will give an algorithm that runs in time $f(n, m, \alpha, \Delta) \cdot L^{O(1)}$, where L is the length of the bit-representation of the input instance. Since we can reduce the input using Lemma 1 first and Δ is upper bounded in terms of n and α this will yield an FPT algorithm for INTEGER QUADRATIC PROGRAMMING parameterized by n and α .

Let r be the dimension of the nullspace of C . Using Cramer's rule (see [16]) we can in polynomial time compute a basis y_1, \dots, y_r for the nullspace of C , such that each y_i is an integer vector and $|y_i|_\infty \leq \Delta^2$. We let Y be the $n \times r$ matrix whose columns are the vectors y_1, \dots, y_r . We will abuse notation and write $y_i \in Y$ to denote that we chose the i 'th column y_i of Y . We will say that a feasible solution x is *deep* if $x + y_i$ and $x - y_i$ are feasible solutions for all $y_i \in Y$. A feasible solution that is not deep is called *shallow*.

Lemma 2. *Let x be a shallow feasible solution to (2). Then there exists a row a_j^T of A and integer b'_j such that $a_j^T x = b'_j$ and $b'_j \in \{b_j - \alpha \cdot n \cdot \Delta^2, b_j\}$. Further, a_j^T is linearly independent from the rows of C .*

Proof. We prove the statement for $y_i \in Y$ such that $x + y_i$ is not a feasible solution to (2). Then there exists a row a_j^T of A such that $a_j^T(x + y_i) > b_j$, and thus

$$b_j - a_j^T y_i < a_j^T x \leq b_j.$$

Thus $a_j^T x = b'_j$ for $b'_j \in \{b_j - \alpha \cdot n \cdot |y_i|_\infty, b_j\}$. Since $|y_i|_\infty \leq \Delta^2$ we have that $b'_j \in \{b_j - \alpha \cdot n \cdot \Delta^2, b_j\}$.

We now show that a_j^T is linearly independent from the rows of C . Suppose not, then there exists a coefficient vector λ such that $\lambda^T C = a_j^T$. But then

$$a_j^T(x + y_i) = \lambda^T C(x + y_i) = \lambda^T Cx + \lambda^T Cy_i = \lambda^T Cx = a_j^T x \leq b_j,$$

which contradicts that $a_j^T(x + y_i) > b_j$. We conclude that a_j^T is linearly independent from the rows of C . The proof for the case when $x^* - y_i$ is not a feasible solution to (2) is symmetric. \square

Lemma 2 suggests the following branching strategy: either all optimal solutions are deep or Lemma 2 applies to some shallow optimal solution x^* . In the latter case the algorithm can branch on the choice of row a_j^T and b_j' and add the equation $a_j^T x = b_j'$ to the set of constraints. This decreases the dimension of the nullspace of C by 1. We are left with handling the case when all optimal solutions are deep.

Lemma 3. *For any pair of vectors $x, y \in \mathbb{R}^n$ and symmetric matrix $Q \in \mathbb{R}^{n \times n}$, the following are equivalent.*

1. $(x + y)^T Q(x + y) \geq x^T Qx$ and $(x - y)^T Q(x - y) \geq x^T Qx$,
2. $-y^T Qy \leq 2x^T Qy \leq y^T Qy$.

Proof. Expanding the inequalities of (1) yields

$$x^T Qx + 2x^T Qy + y^T Qy \geq x^T Qx,$$

$$x^T Qx - 2x^T Qy + y^T Qy \geq x^T Qx.$$

Cancelling the $x^T Qx$ terms and re-organizing yields $2x^T Qy \geq -y^T Qy$ and $2x^T Qy \leq y^T Qy$. Since the left hand side of the inequalities is the same we can combine the two inequalities in a single inequality,

$$-y^T Qy \leq 2x^T Qy \leq y^T Qy,$$

completing the proof. Note that all the manipulations we did on the inequalities are reversible, thus the above argument does indeed prove equivalence and not only the forward direction (1) \rightarrow (2). \square

Lemma 3 suggests a branching strategy to find a deep solution x^* : pick a vector y_i in Y such that $y_i^T Q$ is linearly independent of the rows of C , guess the value z of $2(x^*)^T Qy_i$ and add the linear equation $2(x^*)^T Qy_i = z$ to the set of constraints. In each branch the dimension of the nullspace of C decreases by 1. Thus we are left with the case that all solutions are deep and there is no y_i in Y such that $y_i^T Q$ is linearly independent of the rows of C . We now handle this case.

Lemma 4. *For any deep optimal solution x^* of the IQP (2) and $y_i \in Y$ such that $y_i^T Q$ is linearly dependent of the rows of C , the vectors $x^* + y_i$ and $x^* - y_i$ are also optimal solutions.*

Proof. We prove the statement for $x^* + y_i$. Since x^* is deep it follows that $x^* + y_i$ is feasible, and it remains to lower bound the objective function value of $x^* + y_i$. Since $y_i^T Q$ is linearly dependent of the rows of C there exists a coefficient vector λ^T such that $y_i^T Q = \lambda^T C$. Therefore,

$$\begin{aligned} 2(x^* + y_i)^T Qy_i &= 2y_i^T Q(x^* + y_i) = 2\lambda^T C(x^* + y_i) \\ &= 2\lambda^T Cx^* + 2\lambda^T Cy_i = 2\lambda^T Cx^* = 2y_i^T Qx^* = 2(x^*)^T Qy_i \end{aligned}$$

Since x^* is deep, it follows that both $x^* + y_i$ and $x^* - y_i$ are feasible and therefore cannot have a higher value of the objective function than x^* . Hence, by Lemma 3 we have that $-y_i^T Qy_i \leq 2(x^*)^T Qy_i \leq y_i^T Qy_i$. Since $2(x^* + y_i)^T Qy_i = 2(x^*)^T Qy_i$, we have that

$$-y_i^T Qy_i \leq 2(x^* + y_i)^T Qy_i \leq y_i^T Qy_i.$$

Hence, Lemma 3 applied to $(x^* + y_i)$ implies that

$$(x^*)^T Q x^* = (x^* + y_i - y_i)^T Q (x^* + y_i - y_i) \geq (x^* + y_i)^T Q (x^* + y_i).$$

This means that the objective function value of $x^* + y_i$ is at most that of x^* , hence $x^* + y_i$ is optimal. The proof for $x^* - y_i$ is symmetric. \square

Lemma 4 immediately implies the following corollary.

Corollary 1. *Suppose the IQP (2) has an optimal solution, all optimal solutions to (2) are deep, and for every $y_i \in Y$, $y_i^T Q$ is linearly dependent of the rows of C . Then, for every optimal solution x^* and integer vector $\lambda \in \mathbb{Z}^r$, $x^* + Y\lambda$ is also an optimal solution of (2).*

Proof. Since x^* is optimal and deep, and for every $y_i \in Y$, $y_i^T Q$ is linearly dependent of the rows of C , it follows from Lemma 4 that for every $y_i \in Y$, $x^* + y_i$ and $x^* - y_i$ are also optimal solutions of (2). Since all optimal solutions are deep, $x^* + y_i$ and $x^* - y_i$ are deep. The statement of the corollary now follows by induction on $|\lambda|_1$. \square

We are now ready to state the main structural lemma underlying the algorithm for INTEGER QUADRATIC PROGRAMMING.

Lemma 5. *For any Integer Quadratic Program of the form (2) that has an optimal solution and any x_0 such that $Cx_0 = d$, there exists an optimal solution x^* such that at least one of the following three cases holds.*

1. *There exists a row a_j^T of A and integer $b'_j \in \{b_j - \alpha \cdot n \cdot \Delta^2, b_j\}$ such that $a_j^T x^* = b'_j$, and a_j^T is linearly independent from the rows of C .*
2. *There exists a $y_i \in Y$ such that $y_i^T Q$ is linearly independent of the rows of C and $2y_i^T Q x^* = z$ for $z \in \{-n^2 \Delta^4 \alpha, \dots, n^2 \Delta^4 \alpha\}$.*
3. *$|x^* - x_0|_1 \leq \Delta^2 \cdot n$.*

Proof. Suppose the integer quadratic program (2) has a shallow optimal solution x^* . Then, by Lemma 2 case 1 applies. In the remainder of the proof we assume that all optimal solutions are deep. Suppose now that there is a $y_i \in Y$ such that $y_i^T Q$ is linearly independent of the rows of C . Then, since x^* is a deep optimal solution, both $x^* + y_i$ and $x^* - y_i$ are feasible solutions, so $(x^* + y_i)^T Q (x^* + y_i) \geq (x^*)^T Q x^*$ and $(x^* - y_i)^T Q (x^* - y_i) \geq (x^*)^T Q x^*$. Thus, Lemma 3 implies that $2y_i^T Q x^* = z$ for $z \in \{-y_i^T Q y_i, \dots, y_i^T Q y_i\}$. Furthermore, $y_i^T Q y_i$ is the sum of n^2 terms where each term is a product of an element of y_i (and thus at most Δ^2), another element of y_i , and an element of Q . Thus $z \in \{-n^2 \Delta^4 \alpha, \dots, n^2 \Delta^4 \alpha\}$ and therefore case 2 applies.

Finally, suppose that all $y_i \in Y$ are linearly dependent of the rows of C . Let \hat{x} be an arbitrarily chosen optimal solution to (2). Since $C(x_0 - \hat{x}) = 0$ and Y forms a basis for the nullspace of C there is a coefficient vector $\lambda \in \mathbb{R}^r$ such that $x_0 = \hat{x} + Y\lambda$. Define $\tilde{\lambda}$ from λ by rounding each entry down to the nearest integer. In other words, for every i we set $\tilde{\lambda}_i = \lfloor \lambda_i \rfloor$. Set $x^* = \hat{x} + Y\tilde{\lambda}$. By Corollary 1 we have that x^* is an optimal solution to (2). But $|x^* - x_0|_1 = |Y(\tilde{\lambda} - \lambda)|_1 \leq (\max_i |y_i|_1) \cdot n \leq \Delta(C)^2 \cdot n$, concluding the proof. \square

Theorem 1. *There exists an algorithm that given an instance of INTEGER QUADRATIC PROGRAMMING, runs in time $f(n, \alpha)L^{O(1)}$, and outputs a vector $x \in \mathbb{Z}^n$. If the input IQP has a feasible solution then x is feasible, and if the input IQP is not unbounded, then x is an optimal solution.*

Proof. We assume that input is given on the form (2). The algorithm starts by reducing the input system according to Lemma 1. After this preliminary step the number of constraints m in the IQP is upper bounded by $(2\alpha + 1)^n$. We give a recursive algorithm, based on Lemma 5.

The algorithm begins by computing in polynomial time a basis $Y = y_1, \dots, y_r$ for the nullspace of C , as described in the beginning of Section 2. In particular Y is a matrix of integers, and for every i , $|y_i|_\infty \leq \Delta^2$.

If the dimension of the nullspace of C is 0 the algorithm solves the system $Cx = d$ of linear equations in polynomial time. Let x^* be the (unique) solution to this linear system. If x^* is not an integral vector, or $Ax^* \leq b$ does not hold the algorithm reports that the input IQP has no feasible solution. Otherwise it returns x^* as the optimum.

If C is not full-dimensional, i.e the dimension of the nullspace of C is at least 1, the algorithm proceeds as follows. For each row a_j^T of A and integer $b'_j \in \{b_j - \alpha \cdot n \cdot \Delta^2, b_j\}$ such that a_j^T is linearly independent from the rows of C , the algorithm calls itself recursively on the same instance, but with the equation $a_j^T x = b'_j$ added to the system $Cx = d$. Furthermore, for each $y_i \in Y$ such that $y_i^T Q$ is linearly independent of the rows of C and every integer $z \in \{-n^2 \Delta^4 \alpha, \dots, n^2 \Delta^4 \alpha\}$ the algorithm calls itself recursively on the same instance, but with the equation $2y_i^T Qx = z$ added to the system $Cx = d$. Finally the algorithm computes an arbitrary (not necessarily integral) solution x_0 of the system $Cx = d$, and checks all (integral) vectors within ℓ_1 distance at most $\Delta^2 \cdot n$ from x_0 . The algorithm returns the feasible solution with the smallest objective function value among the ones found in any of the recursive calls, and the search around x_0 .

In the recursive calls, when we add a linear equation to the system $Cx = d$ we extend the matrix C and vector d to incorporate this equation. The algorithm terminates, as in each recursive call the dimension of the nullspace of C is decreased by 1. Further, any feasible solution found in any of the recursive calls is feasible for the original system. Thus, if the algorithm reports a solution then it is feasible. To see that the algorithm reports an optimal solution, consider an optimal solution x^* satisfying the conditions of Lemma 5 applied to the quadratic integer program (2) and vector x_0 . Either x^* will be found in the search around x_0 , or x^* satisfies the linear constraint added in at least one of the recursive calls. In the latter case x^* is an optimal solution to the integer quadratic program of the recursive call, and in this call the algorithm will find a solution with the same objective function value. This concludes the proof of correctness.

We now analyze the running time of the algorithm. First, consider the time it takes to search all integral vectors within ℓ_1 distance at most $\Delta^2 \cdot n$ from x_0 . It is easy to see that there are at most $3^{\Delta^2 \cdot n + n}$ such vectors. We will treat this search as at most $3^{\Delta^2 \cdot n + n}$ recursive calls to instances where the dimension of the nullspace of C is 0. Then the running time in each recursive call is polynomial, and it is sufficient to upper bound the number of leaves in the recursion tree of the algorithm.

We bound the number of leaves of the recursion tree as a function of n – the number of variables, m – the number of rows in A , α – the maximum value of an entry in A or Q , Δ – the maximum absolute value of the determinant of a square submatrix of C , and r – the dimension of the nullspace of C . Notice that the algorithm never changes Q or A , and that the number of variables remains the same throughout the execution of the algorithm. Thus n , m and α do not change throughout the execution. For a fixed value of n , m and α , we let $T(r, \Delta)$ be the maximum number of leaves in the recursion tree of the algorithm when called on an instance with the given value of n , m , α , r and Δ .

In each recursive call the algorithm adds a new row to the matrix C . Let C' be the new matrix after the addition of this row, r' be the dimension of the nullspace of C and Δ' be the maximum value of a determinant of a square submatrix of C' . Since the new added row is linearly independent of the rows of C it follows that $r' = r - 1$ in each of the recursive calls arising from case 1 and case 2 of Lemma 5. The remaining recursive calls are to leaves of the recursion tree.

When the algorithm explores case 1, it guesses a row a_j , for which there are m possibilities, and a value for b'_j , for which there are $\alpha \cdot n \cdot \Delta^2$ possibilities. This generates $m \cdot \alpha \cdot n \cdot \Delta^2$ recursive

calls. In each of these recursive calls a_j is the new row of C' , and so, by the cofactor expansion of the determinant [16], $\Delta' \leq n\alpha\Delta$.

When the algorithm explores case 2, it guesses a vector $y_i \in Y$, and there are at most n possibilities for y_i . For each of these possibilities the algorithm guesses a value for z , for which there are $2n^2\Delta^4\alpha$ possible choices. Thus this generates $2n^3\Delta^4\alpha$ recursive calls. In each of the recursive calls the new row added to C is $2y_i^T Q$. We have that $|y_i|_\infty \leq \Delta^2$. Thus, $|2y_i^T Q|_\infty \leq n \cdot \Delta^2 \cdot \alpha$, and the cofactor expansion of the determinant [16] applied to the new row yields $\Delta' \leq n^2\Delta^3 \cdot \alpha$. It follows that the number of leaves of the recursion tree is governed by the following recurrence.

$$\begin{aligned} T(r, \Delta) &\leq m \cdot \alpha \cdot n \cdot \Delta^2 \cdot T(r-1, n\alpha\Delta) + 2n^3 \cdot \Delta^4 \cdot \alpha \cdot T(r-1, n^2\Delta^3\alpha) + 3^{(\Delta^2+1) \cdot n} \\ &\leq \alpha \cdot m \cdot n^3 \cdot \Delta^4 \cdot T(r-1, n^2\Delta^3\alpha) + 3^{(\Delta^2+1) \cdot n} \end{aligned}$$

The above recurrence is clearly upper bounded by a function of n , m , Δ and α . Since m is upper bounded by $(2\alpha + 1)^n$ from Lemma 1, the theorem follows. \square

3 Optimal Linear Arrangement Parameterized by Vertex Cover

We assume that a vertex cover C of G of size at most k is given as input. The remaining set of vertices $I = V(G) - C$ forms an independent set. Furthermore, I can be partitioned into at most 2^k sets as follows: for each subset S of C we define $I_S = \{v \in I : N(v) = S\}$. For every vertex $v \in I$ we will refer to $N(v)$ as the *type* of v , clearly there are at most 2^k different types.

Let $C = \{c_1, c_2, \dots, c_k\}$. By trying all $k!$ permutations of C we may assume that the optimal solution σ satisfies $\sigma(c_i) < \sigma(c_{i+1})$ for every $1 \leq i \leq k-1$. For every i between 1 and $k-1$ we define the i 'th *gap* of σ to be the set B_i of vertices appearing between c_i and c_{i+1} according to σ . The 0'th gap B_0 is the set of all vertices appearing before c_1 , and the k 'th gap B_k is the set of vertices appearing after c_k . For every gap B_i and type $S \subseteq C$ of vertices we denote by I_S^i the set $B_i \cap I_S$ of vertices of type S appearing in gap i .

We say that an ordering σ is *homogenous* if, for every gap B_i and every type $S \subseteq C$ the vertices of I_S^i appear consecutively in σ . Informally this means that inside the same gap the vertices from different sets I_S and $I_{S'}$ “don't mix”. Fellows et al. [8] show that there always exists an optimal solution that is homogenous.

Lemma 6. [8] *There exists a homogenous optimal linear arrangement of G .*

For every vertex v we define the *force* of v with respect to σ to be

$$\delta(v) = |\{u \in N(v) : \sigma(u) > \sigma(v)\}| - |\{u \in N(v) : \sigma(u) < \sigma(v)\}|.$$

Notice that two vertices of the same type in the same gap have the same force. Fellows et al. [8] in the proof of Lemma 6 show that there exists an optimal solution that is homogenous, and where inside every gap, the vertices are ordered from left to right in non-decreasing order by their force. We will call such an ordering solution *super-homogenous*. As already noted, the existence of a super-homogenous optimal linear arrangement σ follows from the proof of Lemma 6 by Fellows et al. [8].

Lemma 7. [8] *There exists a super-homogenous optimal linear arrangement of G .*

Notice that a super-homogenous linear arrangement σ is completely defined (up to swapping positions of vertices of the same type) by specifying for each i and each type S the size $|I_S^i|$. For each gap i and each type S we introduce a variable $x_S^i \in \mathbb{Z}$ representing $|I_S^i|$. Clearly the variables x_S^i need to satisfy

$$\forall i \leq k, \forall S \subseteq C \quad x_S^i \geq 0 \tag{3}$$

and

$$\forall S \subseteq C \quad \sum_{i=0}^k x_S^i = |I_S|. \quad (4)$$

On the other hand, every assignment to the variables satisfying these (linear) constraints corresponds to a super-homogenous linear arrangement σ of G with $|I_S^i| = x_S^i$ for every type S and gap i .

We now analyze the cost of σ as a function of the variables. The goal is to show that $val(\sigma, G)$ is a quadratic function of the variables with coefficients that are bounded from above by a function of k . The coefficients of this quadratic function are not integral, but *half-integral*, namely integer multiples of $\frac{1}{2}$. The analysis below is somewhat tedious, but quite straightforward. For the analysis it is helpful to re-write $val(\sigma, G)$. For a fixed ordering σ of the vertices we say that an edge uv *flies over* the vertex w if

$$\min(\sigma(u), \sigma(v)) < \sigma(w) < \max(\sigma(u), \sigma(v)).$$

We define the “fly over” relation \sim for edges and vertices, i.e $uv \sim w$ means that uv flies over w . Since an edge uv with $\sigma(u) < \sigma(v)$ flies over the $\sigma(v) - \sigma(u) - 1$ vertices appearing between $\sigma(u)$ and $\sigma(v)$ it follows that

$$val(\sigma, G) = |E(G)| + \sum_{uv \in E(G)} \sum_{\substack{w \in V(G) \\ uv \sim w}} 1.$$

We partition the set of edges of G into several subsets as follows. The set E_C is the set of all edges with both endpoints in C . For every gap $i \in \{0, \dots, k\}$, every $j \in \{1, \dots, k\}$ and every $S \subseteq C$ we denote by $E_{i,j}^S$ the set of edges whose one endpoint is in I_S^i and the other is c_j . Notice that $|E_{i,j}^S|$ is either equal to x_S^i or to 0 depending on whether vertices of type S are adjacent to c_j or not. We have that

$$val(\sigma, G) = |E(G)| + \sum_{\substack{c_i c_j \in E_C \\ c_i c_j \sim w}} \sum_{w \in V(G)} 1 + \sum_{i,j,S} \sum_{uc_j \in E_{i,j}^S} \sum_{\substack{w \in V(G) \\ uc_j \sim w}} 1. \quad (5)$$

Further, for each edge $c_i c_j \in E_C$ (with $i < j$) we have that

$$\sum_{\substack{w \in V(G) \\ c_i c_j \sim w}} 1 = j - i - 1 + \sum_{p=i}^{j-1} \sum_{S \subseteq C} x_S^p.$$

In other words, the first double sum of Equation 5 is a linear function of the variables. Since $|E_C| \leq \binom{k}{2}$ the coefficients of this linear function are integers upper bounded by $\binom{k}{2}$.

We now turn to analyzing the second part of Equation 5. We split the triple sum in three parts as follows.

$$\begin{aligned} & \sum_{i,j,S} \sum_{uc_j \in E_{i,j}^S} \sum_{\substack{w \in V(G) \\ uc_j \sim w}} 1 \\ &= \sum_{i,j,S} \left(\sum_{uc_j \in E_{i,j}^S} \sum_{\substack{w \in C \\ uc_j \sim w}} 1 + \sum_{uc_j \in E_{i,j}^S} \sum_{\substack{w \in I_S^i \\ uc_j \sim w}} 1 + \sum_{uc_j \in E_{i,j}^S} \sum_{\substack{w \in I - I_S^i \\ uc_j \sim w}} 1 \right) \end{aligned} \quad (6)$$

For any fixed i, j and S , any edge $uc_j \in E_{i,j}^S$ the number of vertices $w \in C$ such that $uc_j \sim w$ depends solely on i and j . It follows that

$$\sum_{uc_j \in E_{i,j}^S} \sum_{\substack{w \in C \\ uc_j \sim w}} 1 = f(i, j) \cdot x_S^i$$

for some function f , which is upper bounded by k (since $|C| = k$).

Consider a pair of vertices u, w in I_S^i and a vertex $c_j \in C$ such that vertices of u 's and w 's type are adjacent to c_j . Either the edge uc_j flies over w or the edge wc_j flies over u , but both of these events never happen simultaneously. Therefore,

$$\sum_{uc_j \in E_{i,j}^S} \sum_{\substack{w \in I_S^i \\ uc_j \sim w}} 1 = \binom{x_S^i}{2} = \frac{(x_S^i)^2}{2} - \frac{x_S^i}{2}$$

In other words, this sum is a quadratic function of the variables with coefficients $\frac{1}{2}$ and $-\frac{1}{2}$. Further, if vertices in I_S are not adjacent to c_j this sum is 0.

For the last double sum in Equation 6 consider an edge $uc_j \in E_{i,j}^S$ and vertex $v \in I_{S'}^{i'}$ such that $S' \neq S$ or $i' \neq i$. If uc_j flies over v then all the edges in $E_{i,j}^S$ fly over all the vertices in $I_{S'}^{i'}$. Let $g(i, j, S, i', S')$ be a function that returns 1 if vertices in I_S are adjacent to c_j and all the edges in $E_{i,j}^S$ fly over all the vertices in $I_{S'}^{i'}$. Otherwise $g(i, j, S, i', S')$ returns 0. It follows that

$$\sum_{uc_j \in E_{i,j}^S} \sum_{\substack{w \in I - I_S^i \\ uc_j \sim w}} 1 = x_S^i \cdot \sum_{(i', S') \neq (i, S)} g(i, j, S, i', S') x_{S'}^{i'}.$$

In other words, this sum is a quadratic function of the variables with 0 and 1 as coefficients.

The outer sum of Equation 6 goes over all 2^k choices for S , $k+1$ choices for i and k choices for j . Since the sum of quadratic functions is a quadratic function, this concludes the analysis and proves the following lemma.

Lemma 8. *val(σ, G) is a quadratic function of the variables $\{x_S^i\}$ with half-integral coefficients between $-2^k k^2$ and $2^k k^2$. Furthermore, there is a polynomial time algorithm that given G computes the coefficients.*

For each permutation c_1, \dots, c_k of C we can make an integer quadratic program for finding the best super-homogenous solution to OPTIMAL LINEAR ARRANGEMENT which places the vertices of C in the order c_1, \dots, c_k from left to right. The quadratic program has variable set $\{x_S^i\}$ and constraints as in Equations 3 and 4. The objective function is the one given by Lemma 8, but with every coefficient multiplied by 2. This does not change the set of optimal solutions and makes all the coefficients integral. This quadratic program has at most $2^k \cdot (k+1)$ variables, $2^k \cdot (k+2)$ constraints, and all coefficients are between $-2^{k+1} k^2$ and $2^{k+2} k^2$. Furthermore, since the domain of all variables is bounded the IQP is bounded as well. Thus we can apply Theorem 1 to solve each such IQP in time $f(k) \cdot n$. This proves the main result of this section.

Theorem 2. OPTIMAL LINEAR ARRANGEMENT parameterized by vertex cover is fixed parameter tractable.

4 Conclusions

We have shown that INTEGER QUADRATIC PROGRAMMING is fixed parameter tractable when parameterized by the number n of variables in the IQP and the maximum absolute value α of

the coefficients of the objective function and the constraints, in the case that the input IQP is bounded. With some extra care it is possible to extend our methods to get rid of the requirement that the input IQP is bounded. We used the algorithm for INTEGER QUADRATIC PROGRAMMING to give the first FPT algorithm for OPTIMAL LINEAR ARRANGEMENT parameterized by the size of the smallest vertex cover of the input graph.

We hope that this work opens the gates for further research on the parameterized complexity of non-linear and non-convex optimization problems. There are open problems abound. For example, is INTEGER QUADRATIC PROGRAMMING fixed parameter tractable when parameterized just by the number of variables? What about the parameterization by $n + m$, the number of variables plus the number of constraints? What if we allow quadratic functions in the constraints, not only in the objective function? It is also interesting to investigate the parameterized complexity of QUADRATIC PROGRAMMING, i.e. with real-valued variables rather than integer variables. Finally, there is no reason to stop at quadratic functions. In particular, investigating the parameterized complexity of (integer) mathematical programming with degree-bounded polynomials in the objective function and constraints looks like an exciting research direction. Of course we would not be too surprised if some (or many) of these problems turn out to be intractable, but a priori there is no reason why not many of them should turn out to be FPT as well.

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